(Section 1) Preliminaries to Complex Analysis

(subsection 1) Complex numbers and the complex plane

(subsubsection) Basic Properties

(Def) A complex number takes the form z = x + iy where x and y are real. And i is an imaginary number that satisfies i^{2} = -1. We call x and y the real part and the imaginary part of z, and we write x = Re(z) and y = Im(z). A complex number with zero real part is said to be purely imaginary.

The set of all complex numbers is denoted by \mathbb{C}. The complex numbers can be visualized as the usual Euclidean plane : The complex number z = x + iy \in \mathbb{C} is identified with the point (x,y) \in \mathbb{R}^{2} .

x and y axis of \mathbb{R}^{2} are called the real axis and the imaginary axis.

(Def) If z\_{1} = x\_{1} + iy\_{1} and z\_{2} = x\_{2} + iy\_{2} , then z\_{1} + z\_{2} = (x\_{1} + x\_{2}) + i(y\_{1} + y\_{2}) and z\_{1}z\_{2} = (x\_{1}x\_{2} – y\_{1}y\_{2}) + i (x\_{1}y\_{2} + y\_{1}x\_{2}) .

Commutativity : z\_{1} + z\_{2} = z\_{2} + z\_{1} and z\_{1}z\_{2} = z\_{2}z\_{1} for all z\_{1} , z\_{2} \in \mathbb{C}.

Associativity : (z\_{1} + z\_{2}) + z\_{3} = z\_{1} + (z\_{2} + z\_{3}) ; and (z\_{1}z\_{2}) z\_{3} = z\_{1}(z\_{2}z\_{3}) for z\_{1},z\_{2},z\_{3} \in \mathbb{C}.

Distributivity : z\_{1}(z\_{2}+z\_{3}) = z\_{1}z\_{2} + z\_{1}z\_{3} for all z\_{1}, z\_{2}, z\_{3} \in \mathbb{C}.

(Def) The absolute value of a complex number z = x + iy by |z| = (x^{2} + y^{2})^{1/2} , so that |z| is precisely the distance from the origin to the point (x,y). The triangle inequality |z+w| \le |z| + |w| for all z, w \in \mathbb{C}.

For all z \in \mathbb{C} we have both |Re(z)| \le |z| and |Im(z)| \le |z| , and for all z,w \in \mathbb{C}, ||z|-|w|| \le |z-w| .

The complex conjugate of z = x + iy is defined by \bar{z} = x – iy.

(prop) z is real iff z = \bar{z} and purely imaginary iff z = -\bar{z}.

Re(z) = \frac{z+\bar{z}}{2} and Im(z) = \frac{z-\bar{z}}{2i}.

\frac{1}{z} = \frac{\bar{z}}{|z|^{2}} whenever z \neq 0.

(Def) Polar form z = re^{i\theta} where r >0. Also \theta \in \mathbb{R} is called the argument of z and denoted by arg z.

e^{i\theta} = cos \theta + i sin \theta.

If z = re^{i\theta} and w = se^{i\phi} , then zw = rs e^{i(\theta + \phi)}.

(subsubsection 1.2) Convergence

(Def) A sequence {z\_{1}, z\_{2}, …} of complex numbers is said to converge to w \in \mathbb{C} if \lim\_{n \to \infty| |z\_{n} – w| = 0 , and we write w = \lim\_{n \to \infty} z\_{n}.

A sequence {z\_{n}} is said to be a Cauchy sequence (or simply Cauchy) if |z\_{n} – z\_{m}| \to 0 as n , m \to \infty. In other words, given \epsilon >0 there exists an integer N>0 so that |z\_{n} – z\_{m}| < \epsilon whenever n,m >N.

(Prop) \mathbb{R} is complete : every Cauchy sequence of real numbers converges to a real number. The sequence {z\_{n}} is Cauchy iff the sequences of real and imaginary parts of z\_{n} are.

(Thm 1.1) \mathbb{C} , the complex numbers, is complete.

(subsubsection 1.3.) Sets in the complex plane

If z\_{0} \in \mathbb{C} and r > 0 , we define the open disc D\_{r}(z\_{0}) of radius r centered at z\_{0} to be the set of all complex numbers that are at absolute value strictly less than r from z\_{0}. D\_{r} (z\_{0}) = {z \in \mathbb{C} : |z – z\_{0}| < r}.

The closed disc \bar{D}\_{r}(z\_{0}) = {z \in \mathbb{C} : |z – z\_{0}| < r} and the boundary of either the open or closed disc is the circle C\_{r}(z\_{0}) = {z \in \mathbb{C} : |z-z\_{0}| = r.

Denote unit disc \mathbb{D} = {z \in \mathbb{C} : |z|<1}.

Given a set \Omega \subset \mathbb{C}, a point z\_{0} is an interior point of \Omega if there exists r>0 s.t. D\_{r}(z\_{0}) \subset \Omega.

The interior of \Omega consists of all its interior points. A set \Omega is open if every point in that set is an interior point of \Omega. A set \Omega is closed if its complement \Omega^{c} = \mathbb{C} - \Omega is open. A point z \in \mathbb{C} is said to be a limit point of the set \Omega if there exists a sequence of points z\_{n} \in \Omega s.t. z\_{n} \neq z and \lim\_{n \to \infty} z\_{n} = z. The closure of any set \Omega is the union of \Omega and its limit points, and is often denoted by \bar{\Omega}.

Boundary of a set \Omega is equal to its closure minus its interior, and is often denoted by \partial \Omega. A set \Omega is bounded if there exists M>0 s.t. |z|<M whenever z \in \Omega. If \Omega is bounded, we define its diameter by diam(\Omega) = \sup\_{z,w \in \Omega} |z-w|.

A set \Omega is said to be compact if it is closed and bounded.

(Thm 1.2.) The set \Omega \subset \mathbb{C} is compact iff every sequence {z\_{n}} \subset \Omega has a subsequence that converges to a point in \Omega.

(Def) An open covering of \Omega is a family of open sets {U\_{\alpha}} (not necessarily countable) s.t. \Omega \subset \bigcup\_{\alpha} U\_{\alpha}.

(Thm 1.3.) A set \Omega is compact iff every open covering of \Omega has a finite subcovering.

(prop 1.4.) If \Omega\_{1} \supset \Omega\_{2} \supset \cdots \supset \Omega\_{n} \supset \cdots is a sequence of non-empty compact sets in \mathbb{C} with the property that diam(\Omega\_{n}) \to 0 as n \to \infty, then there exists a unique point w \in \mathbb{C} s.t. w \in \Omega\_{n} for all n.

(Def) An open set \Omega \subset \mathbb{C} is said to be connected if it is not possible to find two disjoint non-empty open sets \Omega\_{1} and \Omega\_{2} s.t. \Omega = \Omega\_{1} \cap \Omega\_{2}.

A connected open set in \mathbb{C} will be called a region. Similarly, a closed set F is connected if one cannot write F = F\_{1} \cap F\_{2} where F\_{1} and F\_{2} are disjoint non-empty closed sets.

(subsection 2) Functions on the complex plane

(subsubsection 2.1.) Continuous functions

(Def) Let f be a function defined on a set \Omega of complex numbers. We say that f is continuous at the point z\_{0} \in \Omega if for every \epsilon >0 there exists \delta>0 s.t. whenever z \in \Omega and |z-z\_{0}| < \delta then |f(z) – f(z\_{0}) | < \epsilon.

(Prop) The function f is said to be continuous on \Omega if it is continuous at every point of \Omega. Sums and products of continuous functions are also continuous.

The function f of the complex argument z = x + iy is continuous iff it is continuous viewed as a function of the two real variables x and y.

(Def) f attains a maximum at the point z\_{0} \in \Omega if |f(z)| \le |f(z\_{0})| for all z \in \Omega, with the inequality reversed for the definition of a minimum.

(Thm 2.1.) A continuous function on a compact set \Omega is bounded and attains a maximum and minimum on \Omega.

(subsubsection 2.2.) Holomorphic functions

(Def) Let \Omega be an open set in \mathbb{C} and f a complex-valued function on \Omega. The function f is holomorphic at the point z\_{0} \in \Omega if the quotient \frac{f(z\_{0} + h) – f(z\_{0})}{h} converges to a limit when h \to 0. The limit of the quotient, when it exists, is denoted by f’(z\_{0}) , and is called the derivative of f at z\_{0} ; f’(z\_{0}) = \lim\_{h \to 0} \frac{f(z\_{0} + h) – f(z\_{0})}{h}

(Def) The function f is said to be holomorphic on \Omega if f is holomorphic at every point of \Omega. If C is a closed subset of \mathbb{C}, we say that f is holomorphic on C if f is holomorphic in some open set containing C . Finally, if f is holomorphic in all of \mathbb{C} we say that f is entire.

The terms regular or complex differentiable or analytic are used instead of holomorphic occasionally.

(Prop) A function f is holomorphic at z\_{0} \in \Omega iff there exists a complex number a s.t. f(z\_{0} + h) – f(z\_{0}) – ah = h \psi(h) , where \psi is a function defined for all small h and \lim\_{h \to 0} \psi(h) = 0. f is continuous whenever it is holomorphic.

(Prop 2.2.) If f and g are holomorphic in \Omega, then

f + g is holomorphic in \Omega and (f + g)’ = f’ + g’.

fg is holomorphic in \Omega and (fg)’ = f’g + fg’.

If g(z\_{0}) \neq 0, then f/g is holomorphic at z\_{0} and (f/g)’ = \frac{f’g – fg’}{g^{2}}.

Moreover, if f : \Omega \to U and g : U \to \mathbb{C} are holomorphic, the chain rule holds

(g \bullet f)’(z) = g’(f(z))f’(z) for all z \in \Omega.

(Def) Let F(x,y) = (u(x,y), v(x,y)) . If F is differentiable, the partial derivative of u and v exist, and the linear transformation J is described in the standard basis of \mathbb{R}^{2} by the Jacobian matrix of F J = J\_{F}(x,y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y } \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}

(Def) \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} and \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x} These are called Cauchy-Riemann equations.

(Def) \frac{\partial}{\partial z} = \frac{1}{2} (\frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} ) and \frac{\partial}{\partial \bar{z}} = \frac{1}{2} (\frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} )

(Prop 2.3.) If f is holomorphic at z\_{0}, then \frac{\partial f}{\partial \bar{z}} (z\_{0}) = 0 and f’(z\_{0}) = \frac{\partial f}{\partial z} (z\_{0}) = 2 \frac{\partial u}{\partial z} (z\_{0}) .

Also, if we write F(x,y) = f(z), then F is differentiable in the sense of real variables, and det J\_{F} (x\_{0}, y\_{0}) = |f’(z\_{0})|^{2}.

(Thm 2.4.) Suppose f = u + iv is a complex-valued function defined on an open set \Omega. If u and v are continuously differentiable and satisfy the Cauchy-Riemann equations on \Omega, then f is holomorphic on \Omega and f’(z) = \partial f / \partial z.

(subsection 2.3.) Power series

(Def) Complex exponential function which is defined for z \in \mathbb{C} by e^{z} = \sum\_{n = 0}^{\infty} \frac{z^{n}}{n!} . This series converges absolutely for every z \in \mathbb{C}.

e^{z} is holomorphic in all of \mathbb{C}. e^{z} is its on derivative.

(Def) A power series is an expansion of the form \sum\_{n = 0}^{\infty} a\_{n} z^{n}.

(Thm 2.5.) Given a power series \sum\_{n = 0}^{\infty} a\_{n} z^{n}, there exists 0 \le R \le \infty s.t.

If |z| < R the series converges absolutely.

If |z| > R the series diverges.

Moreover, if we use the convention that 1/0 = \infty and 1/\infty = 0, then R is given by Hadamard’s formula 1/R = \limsum |a\_{n}|^{1/n}.

The number R is called the radius of convergence of the power series, and the region |z|<R the disc of convergence. In particular, we have R = \infty in the case of exponential function, and R = 1 for the geometric series.

(Def) Trigonometric functions : cos z = \sum\_{n = 0}^{\infty} (-1)^{n} \frac{z^{2n}}{(2n)!} and sin z = \sum\_{n = 0}^{\infty} (-1)^{n} \frac{z^{2n+1}}{(2n+1)!}.

(Prop) cos z = \frac{e^{iz} + e^{-iz}}{2} and sin z = \frac{e^{iz} – e^{-iz}}{2i} These are called the Euler formulas for the cosine and sine functions.

(Thm 2.6.) The power series f(z) = \sum\_{n = 0}^{\infty} a\_{n}z^{n} defines a holomorphic function in its disc of convergence. The derivative of f is also a power series obtained by differentiating term by term the series for f, that is, f’(z) = \sum\_{n = 0}^{\infty} nan\_{n} z^{n-1}.

Moreover, f’ has the same radius of convergence as f.

(Cor 2.7.) A power series is infinitely complex differentiable in its disc of convergence, and the higher derivatives are also power series obtained by termwise differentiation.

(Def) A power series centered at z\_{0} \in \mathbb{C} is an expression of the form f(z) = \sum\_{n = 0}^{\infty} a\_{n}(z – z\_{0})^{n}. The disc of convergence of f is now centered at z\_{0} and its radius is still given by Hadamard’s formula.

(Def) A function f defined on an open set \Omega is said to be analytic (or have a power series expansion) at a point z\_{0} \in \Omega if there exists a power series \sum a\_{n} (z-z\_{0})^{n} centered at z\_{0}, with positive radius of convergence, s.t. f(z) = \sum\_{n = 0}^{\infty} a\_{n}(z-z\_{0})^{n} for all z in a neighborhood of z\_{0}.

If f has a power series expansion at every point in \Omega, we say that f is analytic on \Omega.

(subsection 3) Integration along curves

(Def) A parametrized curve is a function z(t) which maps a closed interval [a,b] \subset \mathbb{R} to the complex plane.

(Def) We say that the parametrized curve is smooth if z’(t) exists and is continuous at [a,b] , and z’(t) \neq 0 for t \in [a,b] . At the points t = a and t = b, the quantities z’(a) and z’(b) are interpreted as the one-sided limits z’(a) = \lim\_{h \to 0 , h >0} \frac{z(a+h) – z(a)}{h} and z’(b) = \lim\_{h \to 0, h < 0} \frac{z(b+h) – z(b)}{h}

(Def) We say that the parametrized curve is piecewise-smooth if z is continuous on [a,b] and if there exist points a = a\_{0} < a\_{1} < \cdots < a\_{n} = b, where z(t) is smooth in the intervals [a\_{k}, a\_{k+1}] .

The right-hand derivative at a\_{k} may differ from the left-hand derivative at a\_{k} for k = 1,…,n-1.

(Def) Two parametrizations, z : [a,b] \to \mathbb{C} and \tilde{z} : [c,d] \to \mathbb{C} are equivalent if there exists a continuously differentiable bijection s \mapsto t(s) from [c,d] \to [a,b] so that t’(s) >0 and \tilde{z}(s) = z(t(s)).

(Def) The family of all paramterizations that are equivalent to z(t) determines a smooth curve \gamma \subset \mathbb{C} , namely the image of [a,b] under z with the orientation given by z as t travels from a to b. define a curve \gamma^{-} obtained from the curve \gamma by reversing the orientation.

(Def) It is clear how to define a piecewise-smooth curve. The points z(a) and z(b) are called the end-points of the curve and are independent on the parametrization. Since \gamma carries an orientation, it is natural to say that \gamma begins at z(a) and ends at z(b).

(DeF) A smooth or piecewise smooth curve is closed if z(a) = z(b) for any of its parametrizations. Finaly, a smooth or piecewise-smooth curve is simple if it is not self-intersecting, that is, z(t) \neq z(s) unless s = t.

(Def) For brevity, call any piecewise-smooth curve a curve.

(Def) The positive orientation(counterclockwise) is the one that is given by the standard parametrization z(t) = z\_{0} + re^{it} , where t \in [0,2\pi], while the negative orientation(clockwise) is given by z(t) = z\_{0} + re^{-it} , where t \in [0,2\pi].

Denote C a general positively oriented circle.

(Def) Given a smooth curve \gamma in \mathbb{C} parametrized by z : [a,b] \to \mathbb{C} , and f a continuous function on \gamma, we define the integral of f along \gamma by \int\_{\gamma} f(z) dz = \int\_{a}^{b} f(z(t)) z’(t) dt.

(Def) If \gamma is piecewise smooth, if z(t) is a piecewise-smooth parametrization as before, then \int\_{\gamma} f(z)dz = \sum\_{k = 0}^{n-1} \int\_{a\_{k}}^{a\_{k+1}} f(z(t))z’(t) dt.

(Def) The length of the smooth curve \gamma is length(\gamma) = \int\_{a}^{b} |z’(t)| dt.

(Prop 3.1.) Integration of continuous functions over curves satisfies the following properties :

It is linear, that is if \alpha, \beta \in \mathbb{C}, then \int\_{\gamma} (\alpha f(z) + \beta g(z)) dz = \alpha \int\_{\gamma} f(z)dz + \beta \int\_{\gamma} g(z) dz.

If \gamma^{-} is \gamma with the reverse orientation, then \int\_{\gamma} f(z) dz = - \int\_{\gamma^{-}} f(z) dz.

One has the inequality |\int\_{\gamma} f(z) dz | \le \sup\_{z \in \gamma} |f(z)} \cdot length(\gamma).

(Def) Suppose f is a function on the open set \Omega. A primitive for f on \Omega is a function F that is holomorphic on \Omega and s.t. F’(z) = f(z) for all z \in \Omega.

(Thm 3.2.) If a continuous function f has a primitive F in \Omega, and \gamma is a curve in \Omega that begins at w\_{1} and ends at w\_{2}, then \int\_{\gamma} f(z) dz = F(w\_{2}) – F(w\_{1}) .

(Cor 3.3.) If \gamma is a closed curve in an open set \Omega, and f is continuous and has a primitive in \Omega, then \int\_{\gamma} f(z)dz = 0.

(Cor 3.4.) IF f is holomorphic in a region \Omega and f’ = 0, then f is constant.

(Def) f(z) = O(g(z)) to mean that there is a constant C > 0 s.t. |f(z)| \le C|g(z)| for z in a neighborhood of the point in question. In addition, we say f(z) = o(g(z)) when |f(z)/g(z) | \to 0. We also write f(z) \simeq g(z) to mean that f(z)/g(z) \to 1.

(Section 2) Cauchy’s Theorem and Its Applications

(subsection 1) Goursat’s Thm

(prop) If f has a primitive in an open set \Omega, then \int\_{\gamma} f(z) dz = 0 for any closed curve \gamma in \Omega. Conversely, if we can show that the above relation holds for some types of curves \gamma, then a primitive will exist.

(Thm 11.1.) If \Omega is an open set in \mathbb{C}, and T \subset \Omega a triangle whose interior is also contained in \Omega, then \int\_{T} f(z) dz = 0 whenever f is holomorphic in \Omega.

(Cor 1.2.) If f is holomorphic in an open set \Omega that contains a rectangle R and its interior, then \int\_{R} f(z) dz = 0.

(subsection 2) Local existence of primitives and Cauchy’s theorem in a disc

(Thm 2.1.) A holomorphic function in an open disc has a primitive in that disc.

(Cauchy’s theorem for a disc) (Thm 2.2.) If f is holomorphic in a disc, then \int\_{\gamma} f(z) dz = 0 for any closed curve \gamma in that disc.

(Cor 2.3.) Suppose f is holomorphic in an open set containing the circle C and its interior, Then \int\_{C} f(z) dz = 0.

(Def) We call a toy contour any closed curve where the notion of interior is obvious, and a construction similar to that in (Thm 2.1.) is possible in a neighborhood of the curve and its interior. Its positive orientation is that for which the interior is to the left as we travel along the toy contour.

(Def) Keyhole \Gamma . It consists of two almost complete circles one large and one small, connected by a narrow corridor. The interior of \Gamma, which we denote by \Gamma\_{int}, is clearly that region enclosed by the curve.

(Prop) For a toy contour \gamma we have that \int\_{\gamma} f(z)dz = 0, whenever f is holomorphic in an open set that contains the contour \gamma and its interior.

(subsection 4) Cauchy’s integral formulas

(Thm 4.1.) Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then f(z) = \frac{1}{2\pi i} \int\_{C} \frac{f(\zeta)}{\zeta - z} d\zeta for any point z \in D.

(Prop) If f is holomorphic in an open set that contains a (positively oriented) rectangle R and its interior, then f(z) = \frac{1}{2\pi i} \int\_{R} \frac{f(\zeta)}{\zeta - z} d\zeta whenever z belongs to the interior of R.

(Cor 4.2.) If f is holomorphic in an open set \Omega, then f has infinitely many complex derivatives in \Omega. Moreover, if C \subset \Omega is a circle whose interior is also contained in \Omega, then f^{(n)} (z) = \frac{n!}{2\pi i} \int\_{C} \frac{f(\zeta)}{(\zeta – z)^{n+1}} d \zeta for all z in the interior of C.

(Cauchy inequalities) (Cor 4.3.) If f is holomorphic in an open set that contains the closure of a disc D centered at z\_{0} and of radius R, then |f^{(n)} (z\_{0})| \le \frac{n!\Vert f \Vert\_{C}}{R^{n}} where \Vert f \Vert\_{C} = \sup\_{z \in C}|f(z)| denotes the supremum of |f| on the boundary circle C.

(Thm 4.4.) Suppose f is holomorphic in an open set \Omega. IF D is a disc centered at z\_{0} and whose closure is contained in \Omega, then f has a power series expansion at z\_{0} f(z) = \sum\_{n = 0}^{\infty} a\_{n} (z-z\_{0})^{n} for all z \in D, and the coefficients are given by a\_{n} = \frac{f^{(n)} (z\_{0})}{n!} for all n \ge 0.

(Liouville’s theorem) (Cor 4.5.) If f is entire and bounded, then f is constant.

(Cor 4.6.) Every non-constant polynomial P(z) = a\_{n}z^{n} + \cdot + a\_{0} with complex coefficients has a root in \mathbb{C}.

(Cor 4.7.) Every polynomial P(z) = a\_{n}z^{n} + \cdots + a\_{0} of degree n \ge 1 has precisely n roots in \mathbb{C}. If these roots are denoted by w\_{1}, …, w\_{n}, then P can be factored as P(z) = a\_{n}(z-w\_{1})(z-w\_{2})\cdot(z-w\_{n}).

(Thm 4.8.) Suppose f is a holomorphic function in a region \Omega that vanishes on a sequence of distinct points with a limit point in \Omega. Then f is identically 0.

(Cor 4.9) Suppose f and g are holomorphic in a region \Omega and f(z) = g(z) for all z in some non-empty open subset of \Omega (or more generally for z in some sequence of distinct points with limit points in \Omega). Then f(z) = g(z) throughout \Omega.

(subsection 5) Further applications

(subsubsection 5.1.) Morera’s Theorem

(Thm 5.1.) Suppose f is a continuous function in the open disc D s.t. for any triangle T contained in D \int\_{T} f(z)dz = 0, then f is holomorphic.

(subsubsection 5.2.) Sequences of holomorphic functions

(Thm 5.2.) If {f\_{n}}\_{n=1}^{\infty} is a sequence of holomorphic functions that converges uniformly to a function f| in every compact subset of \Omega, then f is holomorphic in \Omega.

(Thm 5.3.) Under the hypotheses of the previous theorem, the sequence of derivatives {f’\_{n}}\_{n=1}^{\infty} converges uniformly to f’ on every compact subset of \Omega.

(subsubsection 5.3.) Holomorphic functions defined in terms of integrals

(Thm 5.4.) Let F(z,s) be defined for (z,s) \in \Omega \times [0,1] where \Omega is an open set in \mathbb{C}. Suppose F satisfies the following properties :

F(z,s) is holomorphic in z for each s.

F is continuous on \Omega \times [0,1].

Then the function f defined on \Omega by f(z) = \int\_{0}^{1} F(z,s) ds is holomorphic.

(subsubseciton 5.4.) Schwarz reflection principle

(Def) Let \Omega be an open subset of \mathbb{C} that is symmetric with respect to the real line, that is z \in \Omega iff \bar{z} \in \Omega. Let \Omega^{+} denote the part of \Omega that lies in the upper half-plane and \Omega^{-} that part lies in the lower half-plane.

Let I = \Omega \cap \mathbb{R} so that I denotes the interior of that part of the boundary of \Omega^{+} and \Omega^{-} that lies on the real axis. Then we have \Omega^{+} \cap I \cap \Omega^{-} = \Omega .

(Symmetry Principle) (Thm 5.5.) If f^{+} and f^{-} are holomorphic functions in \Omega^{+} and \Omega^{-} respectively that extend continuously to I and f^{+}(x) = f^{-}(x) for all x \in I, then the function f defined on \Omega by f(z) = \begin{cases} f^{+}(z) & if z \in \Omega^{+} \\ f^{+}(z) = f^{-}(z) & if z \in I \\ f^{-}(z) & if z \in \Omega^{-}\end{cases} is holomorphic on all of \Omega.

(Schwarz reflection principle) (Thm 5.6.) Suppose that f is a holomorphic function in \Omega^{+} that extends continuously to I and s.t. f is real-valued on I. Then there exists a function F holomorphic in all of \Omega s.t. F = f on \Omega^{+}.

(subsubsection 5.5.) Runge’s approximation theorem

(Thm 5.7.) Any function holomorphic in a neighborhood of a compact set K can be approximated uniformly on K by rational functions whose singularities are in K^{c}.

If K^{c} is connected, any function holomorphic in a neighborhood of K can be approximated uniformly on K by polynomials.

(Lem 5.8.) Suppose f is holomorphic in an open set \Omega, and K \subset \Omega is compact. Then, there exists finitely many segments \gamma\_{1}, …, \gamma\_{N} in \Omega – K s.t. f(z) = \sum\_{n = 1}^{N} \frac{1}{2\pi i} \int\_{\gamma\_{n}} \frac{f(\zeta)}{\zeta - z} d \zeta for all z \in K.

(Lem 5.9.) For any line segment \gamma entirely contained in \Omega – K , there exists a sequence of rational functions with singularities on \gamma that approximate the integral \int\_{\gamma}f(\zeta) /(\zeta – z) d\zeta uniformly on K.

(Lem 5.10.) If K^{c} is connected and z\_{0} \notin K, then the function 1/(z-z\_{0}) can be approximated uniformly on K by polynomials.

(Section 3) Meromorphic Functions and the Logarithm

(subsection 1) Zeros and poles

(Def) A point singularity of a function f is a complex number z\_{0} s.t. f is defined in a neighborhood of z\_{0} but not at the point z\_{0} itself. We also call such points isolated singularities.

A complex number z\_{0} is a zero for the holomorphic function f if f(z\_{0}) = 0. In particular, analytic continuation shows that the zeros of a non-trivial holomorphic functions are isolated.

(prop) If f is holomorphic in \Omega and f(z\_{0}) = 0 for some z\_{0} \in \Omega, then there exists an open neighborhood U of z\_{0} s.t. f(z) \neq 0 for all z \in U – {z\_{0}} , unless f is identically zero.

(Thm 1.1.) Suppose that f is holomorphic in a connected open set \Omega, has a zero at a point z\_{0} \in Omega, and does not vanish identically in \Omega. Then there exists a neighborhood U \subset \Omega of z\_{0} , a non-vanishing holomorphic function g on U, and a unique positive integer n s.t. f(z) = (z-z\_{0})^{n}g(z) for all z \in U.

(Def) in the case of above theorem, we say that f has a zero of order n (or multiplicity n) at z\_{0}. If a zero is of order 1, we say that it is simple.

(Def) a deleted neighborhood of z\_{0} to be an open disc centered at z\_{0}, minus the point z\_{0}, that is, the set {z: 0 < |z-z\_{0}| < r} for some r>0. Then, we say that a function f defined in a deleted neighborhood of z\_{0} has a pole at z\_{0}, if the function 1/f, defined to be zero at z\_{0}, is holomorphic in a full neighborhood of z\_{0}.

(Thm 1.2.) If f has a pole at z\_{0} \in \Omega, then in a neighborhood of that point there exists a non-vanishing holomorphic function h and a unique positive integer n s.t. f(z) = (z-z\_{0})^{-n} h(z).

(Def) the integer n is called the order(or multiplicity) of the pole, and describes the rate at which the function grows near z\_{0}. If the pole is of order 1, we say that it is simple.

(Thm 1.3.) If f has a pole of order n at z\_{0}, then f(z) = \frac{a\_{-n}}{(z-z\_{0})^{n}} + \frac{a\_{-n+1}}{(z-z\_{0})^{n-1}} + \cdots + \frac{a\_{-1}}{(z-z\_{0})} + G(z), where G is a holomorphic function in a neighborhood of z\_{0}.

(Def) The sum \frac{a\_{-n}}{(z-z\_{0})^{n}} + \frac{a\_{-n+1}}{(z-z\_{0})^{n-1}} + \cdots + \frac{a\_{-1}}{(z-z\_{0})} is called the principal part of f at the pole z\_{0} , and the coefficient a\_{-1} is the residue of f at that pole. We write res\_{z\_{0}} f = a\_{-1}.

(Thm 1.4.) If f has a pole of order n at z\_{0}, then res\_{z\_{0}} f = \lim\_{z \to z\_{0}} \frac{1}{(n-1)!} (\frac{d}{dz})^{n-1} (z-z\_{0})^{n} f(z).

(subsection 2) The residue formula

(Thm 2.1.) Suppose that f is holomorphic in an open set containing a circle C and its interior, except for a pole at z\_{0} inside C. Then \int\_{C} f(z)dz = 2\pi i res\_{z\_{0}} f.

(Cor 2.2.) Suppose that f is holomorphic in an open set containing a circle C and its interior, except for poles at the points z\_{1}, …, z\_{N} inside C. Then \int\_{C} f(z) dz = 2\pi i \sum\_{k = 1}^{N} res\_{z\_{k}} f.

(Cor 2.3.) Suppose that f is holomorphic in an open set containing a toy contour \gamma and its interior, except for poles at the points z\_{1},, …, z\_{N} inside \gamma. Then \int\_{\gamma} f(z) dz = 2\pi i \sum\_{k = 1}^{N} res\_{z\_{k}} f.

(Def) The identity \int\_{\gamma} f(z) dz = 2\pi i \sum\_{k = 1}^{N} res\_{z\_{k}} f is referred to as the residue formula.

(subsection 3) Singularities and meromorphic functions

(Def) Let f be a function holomorphic in an open set \Omega except possibly at one point z\_{0} in \Omega. If we can define f at z\_{0} in such a way that f becomes holomorphic in all of \Omega, we say that z\_{0} is a removable singularity for f.

(Riemann’s theorem on removable singularities) (Thm 3.1.) Suppose that f is holomorphic in an open set \Omega except possibly at a point z\_{0} in \Omega. If f is bounded on \Omega – {z\_{0}} , then z\_{0} is a removable singularity.

(Cor 3.2.) Suppose that f has an isolated singularity at the point z\_{0}. Then z\_{0} is a pole of f iff |f(z)| \to \infty as z \to z\_{0}.

(Def) Isolated singularities belong to one of three categories : Removable singularities (f bounded near z\_{0}), Pole singularities (|f(z)| \to \infty as z \to z\_{0}) , Essential singularities. Any singularity that is not removable or a pole is defined to be an essential singularity.

(Casorati-Weierstrass) (Thm 3.3.) Suppose f is holomorphic in the punctured disc D\_{r}(z\_{0}) – {z\_{0}} and has an essential singularity at z\_{0}. Then, the image of D\_{r}(z\_{0}) – {z\_{0}} under f is dense in the complex plane.

(Def) A function f on an open set \Omega is meromorphic if there exists a sequence of points {z\_{0}, z\_{1},z\_{2},…} that has no limit poits in \Omega, and such that

the function f is holomorphic in \Omega – {z\_{0}, z\_{1}, z\_{2}, …}, and

f has poles at the points {z\_{0}, z\_{1}, z\_{2},…} .

If f is holomorphic for all large values of z, we consider F(z) = f(1/z), which is now holomorphic in a deleted neighborhood of the origin. We say that f has a pole at infinity if F has a pole at the origin. Similarly, we can speak of f having an essential singularity at infinity, or a removable singularity (hence holomorphic) at infinity in terms of the corresponding behavior of F at 0. A meromorphic function in the complex plane that is either holomorphic at infty or has a pole at infinity is said to be meromorphic in the extended complex plane.

(Thm 3.4.) The meromorphic functions in the extended complex plane are the rational functions.

(Def) Consider the Euclidean space \mathbb{R}^{3} with coordinates (X,Y,Z) where the XY-plane is identified with \mathbb{C} . We denote by \mathbb{S} the sphere centered at (0,0,1/2) and of radius 1/2; Also, we let \mathcal{N} = (0,0,1) denote the north pole of the sphere.

(Def) Given any point W = (X,Y,Z) on \mathbb{S} different from the north pole, the line joining \mathcal{N} and W intersects the XY-plane in a single point which we denote by w = x + iy; w is called the stereographic projection of W. This geometric construction gives a bijective correspondence between points on the punctured sphere \mathbb{S} – {\mathcal{N}} and the complex plane; It is described analytically by the formulas x = \frac{X}{1-Z} and y = \frac{Y}{1-Z} giving w in terms of W, and X = \frac{x}{x^{2} + y^{2} + 1}, Y = \frac{y}{x^{2} + y^{2} + 1} , and Z = \frac{x^2 + y^{2} }{x^{2} + y^{2} + 1} giving W in terms of w.

(Def) Identifying infinity with the point \mathcal{N} on \mathbb{S}, we see that the extended complex plane can be visualized as the full two-dimensional sphere \mathbb{S}; this is the Riemann sphere. Since this construction takes the unbounded set \mathbb{C} into the compact set \mathbb{S} by adding one point, the Riemann sphere is sometimes called the one-point compactification of \mathbb{C}.

(subsection 4) The argument principle and applications

(Argument principle) (Thm 4.1.) Suppose f is meromorphic in an open set containing a circle C and its interior. If f has no poles and never vanishes on C, then \frac{1}{2\pi i} \int\_{C} \frac{f’(z)}{f(z)}dz = (number of zeros of f inside C) minus (number of poles of f inside C), where the zeros and poles are counted with their multiplicities.

(Cor 4.2.) The above theorem holds for toy contours.

(Rouche’s theorem) (Thm 4.3.) Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If |f(z)| > |g(z)| for all z \in C, then f and f+g have the same number of zeros inside the circle C.

(Def) A mapping is said to be open if it maps open sets to open sets.

(Open mapping theorem) (Thm 4.4.) If f is holomorphic and non-constant in a region \Omega, then f is open.

(Def) Refer to the maximum of a holomorphic function f in an open set \Omega as the maximum of its absolute value |f| in \Omega.

(Maximum modulus principle) (Thm 4.5.) If f is a non-constant holomorphic function in a region \Omega, then f cannot attain a maximum in \Omega.

(Cor 4.6.) Suppose that \Omega is a region with compact closure \bar{Omega}. If f is holomorphic on \Omega and continuous on \bar{\Omega} then \sup\_{z \in \Omega} |f(z)| \le \sup\_{z \in \bar{\Omega} - \Omega} |f(z)|.

(subsection 5) Homotopies and simply connected domains

(Def) Let \gamma\_{0} and \gamma\_{1} be two curves in an open set \Omega with common end-points. So if \gamma\_{0}(t) and \gamma\_{1}(t) are two parametrizations defined on [a,b], we have \gamma\_{0}(a) = \gamma\_{1}(a) = \alpha and \gamma\_{0}(b) = \gamma\_{1}(b) = \beta. These two curves are said to be homotopic in \Omega if for each 0 \le s \le 1 there exists a curve \gamma\_{s} \subset \Omega, parametrized by \gamma\_{s}(t) defined on [a,b] , s.t. for every s \gamma\_{s}(a) = \alpha and \gamma\_{s}(b) = \beta, and for all t \in [a,b] \gamma\_{s}(t)|\_{s = 0} = \gamma\_{0}(t) and \gamma\_{s}(t)|\_{s=1} = \gamma\_{1}(t). Moreover, \gamma\_{s}(t) should be jointly continuous in s \in [0,1] and t \in [a,b].

(Thm 5.1.) If f is holomorphic in \Omega, then \int\_{\gamma\_{0}} f(z) dz = \int\_{\gamma\_{1}} f(z) dz whenever the two curves \gamma\_{0} and \gamma\_{1} are homotopic in \Omega.

(Def) A region \Omega in the complex plane is simply connected if any two pair of curves in \Omega with the same end-points are homotopic.

(Thm 5.2.) Any holomorphic function in a simply connected domain has a primitive.

(Cor 5.3.) If f is holomorphic in the simply connected region \Omega, then \int\_{\gamma} f(z)dz = 0 for any closed curve \gamma in \Omega.

(subsection 6) The complex logarithm

(Def) To make sense of the logarithm as a single-valued function, we must restrict the set on which we define it. This is the so-called choice of a branch or sheet of the logarithm.

(Thm 6.1.) Suppose that \Omega is simply connected with 1 \in \Omega, and 0 \notin \Omega. Then in \Omega there is a branch of the logarithm F(z) = log\_{\Omega}(z) so that

F is holomorphic in \Omega,

e^{F(z)} = z for all z \in \Omega,

F(r) = log r whenever r is a real number and near 1.

In other words, each branch \log\_{\Omega}(z) is an extension of the standard logarithm defined for positive numbers.

(Def) In the slit plane \Omega = \mathbb{C} – {(-\infty, 0]} we have the principle branch of the logarithm log z = log r + i \theta where z = re^{i\theta} with |\theta| < \pi.

(Thm 6.2.) If f is a nowhere vanishing holomorphic function in a simply connected region \Omega, then there exists a holomorphic function g on \Omega s.t. g(z) = e^{g(z)} .

The function g(z) in the theorem can be denoted by log f(z), and determines a branch of that logarithm.

(subsection 7) Fourier series and harmonic functions

(Thm 7.1.) The coefficients of the power series expansion of f are given by a\_{n} = \frac{1}{2\pi r^{n}} \int\_{0}^{2\pi} f(z\_{0} + re^{i\theta}) e^{-in\theta} d\theta for all n \ge 0 and 0 < r < R. Moreover, 0 = \frac{1}{2\pi r^{n}} \int\_{0}^{2 \pi} f(z\_{0} + re^{i\theta}) e^{-in\theta} d \theta whenever n < 0.

(Mean-value property) (Cor 7.2.) If f is holomorphic in a disc D\_{R}(z\_{0}), then f(z\_{0}) = \frac{1}{2\pi} \int\_{0}^{2\pi} f(z\_{0} + re^{i\theta}) d\theta , for any 0 < r < R.

(Cor 7.3.) If f is holomorphic in a disc D\_{R}(z\_{0}) , and u = Re(f), then u(z\_{0}) = \frac{1}{2\pi} \int\_{0}^{2\pi} u(z\_{0} + re^{i\theta}) d\theta, for any 0 < r < R.

(Section 4) The Fourier Transform

(Def) If f is a function on \mathbb{R} that satisfies appropriate regularity and decay conditions, then its Fourier transform is defined by \hat(\xi) = \int\_{-\infty}^{\infty} f(x) e^{-2\pi ix \xi} dx , \xi \in \mathbb{R}, and its counterpart, the Fourier inversion formula, holds f(x) = = \int\_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi ix \xi} d\xi

(Def) The condition, given by the Paley-Wiener theorem, is that there be a holomorphic extension of f to \mathbb{C} that satisfies the growth condition |f(z)| \le Ae^{2\piM|z|} for some constant A>0. Functions satisfying this condition are said to be of exponential type.

(Def) A function f is of moderate decrease if f is continuous and there exists A >0 so that |f(x)| \le A/(1+x^{2}) for all x \in \mathbb{R} . A more restrictive condition is that f \in S, the Schwarz space of testing functions, which also implies that \hat{F} belongs to S.

(subsection 1) The class \mathfrak{F}

(Def) For each a >0 we denote by \mathfrak{F}\_{a} the class of all functions f that satisfy the following two conditions :

The function f is holomorphic in the horizontal strip S\_{a} = {z \in \mathbb{C} : |Im(z)| <a } .

There exists a constant A >0 s.t. |f(x+iy)| \le \frac{A}{1+x^{2}} for all x \in \mathbb{R} and |y|<a.

(Def) Denote by \mathfrak{F} the class of all functions that belong to \mathfrak{F}\_{a} for some a.

(subsection 2) Action of the Fourier transform on \mathfrak{F}

(Thm 2.1.) If f belongs to the class \mathfrak{F}\_{a} for some a > 0 , then |\hat{f}(\xi)| \le Be^{-2\pib |\xi|} for any 0 \le b < a.

(Thm 2.2.) If f \in \mathfrak{F}, then the Fourier inversion holds, namely f(x) = \int\_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi ix \xi}d \xi for all x \in \mathbb{R}.

(Lem 2.3.) If A is positive and B is real, then \int\_{0}^{\infty} e^{-(A+iB)\xi} d\xi = \frac{1}{A+iB}.

(Thm 2.4.) If f \in \mathfrak{F}, then \sum\_{n \in \mathbb{Z}}f(n) = \sum\_{n \in \mathbb{Z}} \hat{f}(n).

(subsection 3) Paley-Wiener theorem

(Thm 3.1.) Suppose \hat{f} satisfies the decay condition |\hat{f}(\xi)} \le Ae^{-2\pi a |\xi|} for some constants a, A >0. Then f(x) is the restriction to \mathbb{R} of a function f(z) holomorphic in the strip S\_{b} = {z \in \mathbb{C} : |Im(z)|<b}, for any 0 < b < a.

(Cor 3.2.) If \hat{f} (\xi) = O(e^{-2\pi a |\xi|} ) for some a >0 , and f vanishes in a non-empty open interval, then f = 0.

(Thm 3.3.) Suppose f is continuous and of moderate decrease on \mathbb{R}. Then f has an extension to the complex plane that is entire with |f(z)| \le Ae^{2\piM|z|} for some A>0, iff \hat{f} is supported in the interval [-M,M].

(Thm 3.4.) Suppose F is a holomorphic function in the sector S = {z : -\pi/4 < arg z < \pi/4 } that is continuous on the closure of S. Assume |F(z)| \le 1 on the boundary of the sector, and that there are constants C, c>0 s.t. |F(z)| \le Ce^{c|z|} for all z in the sector. Then |F(z)| \le 1 for all z \in S.

(Thm 3.5.) Suppose f and \hat{f} have moderate decrease. Then \hat{f} (\xi) = 0 for all \xi < 0 iff f can be extended to a continuous and bounded function in the closed upper half-plane {z = x + iy : y \ge 0} with f holomorphic in the interior.

(section 5) Entire Functions

(subsection 1) Jensen’s formula

(Thm 1.1.) Let \Omega be an open set that contains the closure of a disc D\_{R} and suppose that f is holomorphic in \Omega, f(0) \neq 0, and f vanishes nowhere on the circle C\_{R}. If z\_{1},…,z\_{N} denote the zeros of f inside the disc(counted with multiplicities). then log |f(0)| = \sum\_{k = 1}^{N} log(\frac{|z\_{k}|}{R}) + \frac{1}{2\pi} \int\_{0}^{2\pi} log |f(Re^{i\theta})| d\theta.

(Lem 1.2.) If z\_{1},…,z\_{N} are the zeros of f inside the disc D\_{R}, then \int\_{0}^{R} n(r) \frac{dr}{r} = \sum\_{k=1}^{N} log |\frac{R}{z\_{k}}|.

(subsection 2) Functions of finite order

(Def) Let f be an entire function. If there exists a positive number \rho and constants A,B >0 s.t. |f(z)| \le Ae^{B|z|^{\rho}} for all z \in \mathbb{C}, then we say that f has an order of growth \le \rho. We deine the order of growth of f as \pho\_{f} = \inf \tho, where the infimum is over all \pho >0 s.t. f has an order of growth \le \rho.

(Thm 2.1.) If f is an entire function that has an order of growth \le \rho, then:

n(r) \le Cr^{\rho} for some C>0 and all sufficiently large r.

If z\_{1},z\_{2},… denote the zeros of f, with z\_{k} \neq 0, then for all s > \rho we have \sum\_{k=1}^{\infty} \frac{1}{|z\_{k}|^{s}} < \infty.

(subsection 3) Infinite products

(subsubsection 3.1.) Generalities

(Def) Given a sequence {a\_{n}}\_{n=1}^{\infty} of complex numbers, we say that the product \prod\_{n=1}^{\infty} (1+a\_{n}) converges if the limit \lim\_{N \to \infty} \prod\_{n=1}^{\infty}(1+a\_{n}) of the partial products exists.

(Prop 3.1.) If \sum |a\_{n}| < \infty, then the product \prod\_{n=1}^{\infty} (1+a\_{n}) converges. Moreover, the product converges to 0 iff one of its factors is 0.

(prop 3.2.) Suppose {F\_{n}} is a sequence of holomorphic functions on the open set \Omega. If there exist constatns c\_{n} >0 s.t. \sum c\_{n} < \infty and |F\_{n}(z) – 1| \le c\_{n} for all z \in \Omega, then

The product \prod\_{n =1}^{\infty} F\_{n}(z) converges unifotmly in \Omega to a holomorphic function F(z).

If F\_{n}(z) does not vanish for any n, then \frac{F’(z)}{F(z)} = \sum\_{n=1}^{\infty} \frac{F’\_{n}(z)}{F\_{n}(z)}.

(subsubseciton 3.2.) Example : The product formula for the sine function

(prop) \frac{sin \pi z}{\pi} = z \prod\_{n=1}^{\infty} (1-\frac{z^{2}}{n^{2}}).

(subsection 4) Weierstrass infinite products

(Thm 4.1.) Given any sequence {a\_{n}} of complex numbers with |a\_{n}| \to \infty as n \to \infty, there exists an entire function f that vanishes at all z = a\_{n} and nowhere else. Any other such entire function is of the form f(z) e^{g(z)}} , where g is entire.

(Def) For each integer k \ge 0 we define canonical factors by E\_{0}(z) = 1-z and E\_{k}(z) = (1-z)e^{z + z^{2}/2 + \cdot + z^{k}/k}, for k \ge 1. The integer k is called the degree of the canonical factor.

(Lem 4.2.) If |z| \le 1/2, then |1-E\_{k}(z)| \le c|z|^{k+1} for some c>0.

(subsection 5) Hadamard’s factorization theorem

(Thm 5.1.) Suppose f is entire and has growth order \rho\_{0}. Let k be the integer so that k \le \rho\_{0} < k+1. If a\_{1}, a\_{2}, … denote the (non-zero) zeros of f, then f(z) = e^{P(z)} z^{m} \prod\_{n=1}^{\infty} E\_{k}(z/a\_{n}) , where P is a polynomial of degree \le k, and m is the order of the zero of f at z = 0.

(Lem 5.2.) The canonical products satisfy |E\_{k}(z)| \ge e^{-c|z|^{k+1}} if |z| \le 1/2 and |E\_{k}(z)} \ge |1-z| e^{-c’|z|^{k}} if |z| \ge 1/2.

(Lem 5.3.) For any s with \rho\_{0} < s < k+1, we have |\prod\_{n=1}^{\infty} E\_{k} (z/a\_{n})| \ge e^{-c|z|^{s}}, except possibly when z belongs to the union of the discs centered at a\_{n} of radius |a\_{n}|^{-k-1}, for n = 1,2,3,…

(Cor 5.3.) There exists a sequence of radii, r\_{1},r\_{2},…, with r\_{m} \to \infty, s.t. |\prod\_{n=1}^{\infty} E\_{k}(z/a\_{n})| \ge e^{-c|z|^{s}} for |z| = r\_{m}.

(Lem 5.5.) Suppose g is entire and u = Re(g) satisfies u(z) \le Cr^{s} whenever |z| = r for a sequence of positive real numbers r that tends to infinity. Then g is polynomial of degree \le s.

(Section 6) The Gamma and Zeta functions

(subsection 1) The gamma function

(Def) For s >0, the gamma function is defined by \Gamma(s) = \int\_{0}^{\infty} e^{-t}t^{s-1} dt.

(Prop 1.1.) The gamma function extends to an analytic function in the half-plane Re(s) >0, and is still given there by the integral formula as defined.

(subsubsection 1.1.) Analytic continuation

(Lem 1.2.) If Re(s) >0, then \Gamma(s+1) = s \Gamma(s) as a consequence \Gamma(n+1) = n! for n = 0,1,2,…

(Thm 1.3.) The function \Gamma(s) initially defined for Re(s) >0 has an analytic continuation to a meromorphic function on \mathbb{C} whose only singularities are simple poles at the negative integers s = 0, -1, …. The residue of \Gamma at s = -n is (-1)^{n}/n!.

(subsection 1.2.) Further properties of \Gamma

(Thm 1.4.) For all s \in \mathbb{C}, \Gamma(s) \Gamma(1-s) = \frac{\pi}{sin \pi s}.

(Lem 1.5.) For 0<a<1, \int\_{0}^{\infty}\frac{v^{a-1}}{1+v} dv = \frac{\pi}{sin \pi a} .

(Thm 1.6.) The function \Gamma has the following properties : 1/\Gamma(s) is an entire function of s with simple zeros at s = 0,-1,-2,… and it vanishes nowhere else.

1/\Gamma(s) has growth |\frac{1}{\gamma(s)}| \le c\_{1} e^{c\_{2} |s| log|s|}. Therefore, 1/\Gamma is of order 1 in the sense that for every \epsilon >0, there exists a bound c(\epsilon) so that |\frac{1}{\Gamma(s)}| \le c(\epsilon)e^{c\_{2}}|s|^{1+\epsilon}.

(Thm 1.7.) For all s \in \mathbb{C}, \frac{1}{\Gamma(s)} = e^{\gamma s} s \prod\_{n = 1}^{\infty} (1+\frac{s}{n}) e^{-s/n}. The real number \gamma, which is known as Euler’s constant, is defined by \gamma = \lim\_{N \to \infty} \sum\_{n = 1}^{N} \frac{1}{n} – log N.

(subsection 2) The zeta function

(Def) The Riemann zeta function is initially defined for real s >1 by the convergent series \zeta(s) = \sum\_{n=1}^{\infty} \frac{1}{n^{s}}.

(subsubsection 2.1.) Functional equation and analytic continuation

(Prop 2.1.) The series defining \zeta(s) converges for Re(s) >1, and the function \zeta is holomorphic in this half-plane.

(Def) Consider the theta function, which is defined for real t>0 by \vartheta (t) = \sum\_{n = -\infty}^{\infty} e^{-\pi n^{2} t}.

(Thm 2.2.) If Re(s) >1, then \pi^{-s/2} \Gamma(s/2) \zeta(s) = \frac{1}{2} \int\_{0}^{\infty} u^{(s/2)-1} [\vartheta(u) -1] du.

(Def) The modification of the \zeta function called the xi function, which makes the former appear more symmetric. It is defined for Re(s) >1 by \xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s).

(Thm 2.3.) The function \xi is holomorphic for Re(s) >1 and has an analytic continuation to all of \mathbb{C} as a meromorphic function with simple poles at s = 0 and s = 1. Moreover, \xi(s) = \xi (1-s) for all s \in \mathbb{C}.

(Thm 2.4.) The zeta function has a meromorphic continuation into the entire complex plane, whose only singularity is a simple pole at s = 1.

(Prop 2.5.) There are a sequence of entire functions {\delta\_{n}(s)}\_{n=1}^{\infty} that satisfy the estimate |\delta\_{n}(s)| \ge |s|/n^{\sigma +1} , where s = \sigma + it , and such that \sum\_{1 \le n < N} \frac{1}{n^{s}}-\infty\_{1}^{N} \frac{dx}{x^{s}} = \sum\_{1 \le n < N} \delta\_{n}(s), whenever N is an integer >1.

(Cor 2.6.) For Re(s) >0 we have \zeta(s) - \frac{1}{s-1} = H(s) , where H(s) = \sum\_{n = 1}^{\infty} \delta\_{n}(s) is holomorphic in the half-plane Re(s) >0.

(Prop 2.7.) Suppose s = \sigma + it with \sigma, t \in \mathbb{R}. Then for each \sigma\_{0}, 0 \le \sigma\_{0} \le 1 , and every \epsilon >0, there exists a constant c\_{\epsilon} so that |\zeta(s)| \le c\_{\epsilon} |t|^{1-\sigma\_{0} + \epsilon}, if \sigma\_{0} \le \sigma and |t| \ge 1.

|\zeta’(s)| \le c\_{\epsilon} |t|^{\epsilon}, if 1 \le \sigma and |t| \ge 1.

(section 7) The zeta function and prime number theorem

(subsection 1) Zeros of zeta function

(prop) \zeta(s) = \prod\_{p} \frac{1}{1-p^{-s}} .

\zeta(s) = \pi^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \zeta(1-s)

(Thm 1.1.) The only zeros of \zeta outside the strip 0\le Re(s) \le 1 are at the negative even integers, -2,-4,-6,… The region that remains to be studied is called the critical strip.

(Def) Riemann Hypothesis : The zeros of \zeta(s) in the critical strip lie on the line Re(s) = 1/2.

the zeros of \zeta located outside the critical strip are sometimes called the trivial zeros of the zeta function.

(Thm 1.2.) The zeta function has no zeros on the line Re(s) = 1.

(Lem 1.3.) If Re(s) >1, then log \zeta(s) = \sum\_{p,m} \frac{p^{-ms}}{m} = \sum\_{n=1}^{\infty}c\_{n}n^{-s} for some c\_{n} \ge 0.

(Cor 1.5.) If \sigma >1 and t is real, then log|\zeta^{3}(\sigma) \zeta^{4}(\sigma + it) \zeta(\sigma + 2it)| \ge 0.

(subsection 1.1.) Estimates for 1/\zeta(s)

(prop 1.6.) For every \epsilon >0, we have 1/|\zeta(s)| \le c\_{\epsilon} |t|^{\epsilon} when s = \sigma + it, \sigma \ge 1, and |t| \ge 1.

(section 2) Reduction to the functions \psi and \psi\_{1}

(Def) Tchebychev’s \psi-function is defined by \psi(x) = \sum\_{p^{m} \le x} log p.

(Prop 2.1.) If \psi(x) \sim x as x \to \infty, then \pi(x) \sim x/log x as x \to \infty.

(Prop 2.2.) If \psi\_{1}(x) \sim x^{2}/2 as x \to \infty, then \psi(x) \sim x as x \to \infty, and therefore \pi(x) \sim x/log x as x \to \infty.

(Prop 2.3.) For all c > 1 \psi\_{1} (x) = \frac{1}{2\pi i} \int\_{c-i\infty}^{c + i\infty} \frac{x^{s+1}}{s(s+1)} (-\frac{\zeta’(s)}{\zeta(s)}) ds.

(Lem 2.4.) If c >0, then \frac{1}{2\pi i} \int\_{c-i\infty}^{c + i\infty}\frac{a^{s}}{s(s+1)} ds = \begin{cases}0 & if 0 < a \le 1 \\ 1-1/a & if 1\le a. \end{cases}

Here, the integral is over the vertical line Re(s) = c.

(subsubsection 2.1.) Proof of the asymptotic for \psi\_{1}

(prop) \psi\_{1}(x) \sim x^{2}/2 as x \infty.

(subsubsection) Note on interchanging double sums. if {a\_{kl}\_{1\le k,l < \infty} is a sequence of complex numbers indexed by \mathbb{N} \times \mathbb{N}, s.t. \sum\_{k=1}^{\infty} (\sum\_{l=1}^{\infty} |a\_{kl}|) <\infty, then

The double sum A = \sum\_{k=1}^{\infty} (\sum\_{l=1}^{\infty} a\_{kl} ) summed in this order converges, and we may in fact interchange the order of summation, so that A = \sum\_{k=1}^{\infty} \sum\_{l=1}^{\infty} a\_{kl} = \sum\_{l=1}^{\infty} \sum\_{k=1}^{\infty} a\_{kl} .

Given \epsilon >0, there is a positive integer N so that for all K,L >N we have |A-\sum\_{k=1}^{K} \sum\_{l=1}^{L} a\_{kl} | <\epsilon.

If m \mapsto (k(m), l(m)) is a bijection from \mathbb{N} to \mathbb{N} \times \mathbb{N}, and if we write c\_{m} = a\_{k(m)l(m)}, then A = \sum\_{k=1}^{\infty} c\_{k}.

(Def) A bijective holomorphic function f : U \to V is called a conformal map or biholomorphism. Given such a mapping f, we say that U and V are conformally equivalent or simply biholomorphic.

(Prop 1.1.) If f : U \to V is holomorphic and injective, then f’(z) \neq 0 for all z \in U. In particular, the inverse of f defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic.

(subsubseciton 1.1.) The disc and upper half-plane

(Def) The upper half-plane, which we denote by \mathbb{H}, consists of those complex numbers with positive imaginary part; that is, \mathbb{H} = {z \in \mathbb{C} : Im(z) >0}.

(Thm 1.2.) The map F : \mathbb{H} \to \mathbb{D} : F(z) = \frac{i-z}{i+z} is a conformal map with inverse G : \mathbb{D} \to \mathbb{H} : G(w) = i\frac{1-w}{1+w}.

(Def) Mappings of the form z \mapsto \frac{az+b}{cz+d} where a,b,c, and d are complex numbers, and where the denominator is assumed not to be multiple of the numerator, are usualy referred to as fractional linear transformations.

(subsubsection 1.2.) Further examples

(Def) For any non-zero complex number c, the map f : z \mapsto cz is a conformal map from the complex plane to itself, whose inverse is simply g: w\mapsto c^{-1}w . If c has modulus 1, so that c = e^{i\varphi} for some real \varphi, then f is a rotation by \varphi. If c >0, then f corresponds to a dilation.

(subsubsection 1.3.) The Dirichlet problem in a strip

(Def) The Dirichlet problem in the open set \Omega consists of solving \begin{cases} \Delta u = 0 & in \Omega \\ u = f & on \partial \Omega, \end{cases} where \Delta denotes the Laplacial \partial^{2}/\partial x^{2} + \partial^{2}/\partial y^{2} , and f is a given function on the boundary of \Omega .

(Lem 1.3.) Let V and U be open sets in \mathbb{C} and F : V \to U a holomorphic function. If u : U \to \mathbb{C} is a harmonic function, then u \bullet F is harmonic on V.

(subsection 2) The Schwarz lemma ; automorphisms of the disc and upper half-plane

(Lem 2.1.) Let f : \mathbb{D} \to \mathbb{D} be holomorphic with f(0) = 0. Then |f(z)| \le |z| for all z \in \mathbb{D}.

If for some z\_{0} \neq 0 we have |f(z\_{0})| = |z\_{0}| , then f is a rotarion.

|f’(0)| \le 1, and if equality holds, then f is a rotation.

(subsubsection 2.1.) Automorphisms of the disc

(Def) A conformal map from an open set \Omega to itself is called an automorphism of \Omega. The set of all automorphisms of \Omega is denoted by Aut(\Omega), and carries the structure of a group.

if f and g are automorphisms of \Omega, then f \bullet g is also an automorphism.

(Thm 2.2.) If f is an automorphism of the disc, then there exist \theta \in \mathbb{R} and \alpha \in \mathbb{D} s.t. f(z) = e^{i\theta} \frac{\alpha - z}{1-\bar{\alpha} z}.

(Cor 2.3.) The only automorphisms of the unit disc that fix the origin are the rotations.

(subsubsection 2.2.) Automorphisms of the upper half-plane

(prop ) Aut(\mathbb{D}) and Aut(\mathbb{H}) are the same.

Aut(\mathbb{H}) consists of all maps z \mapsto \frac{az + b}{cz + d} where a,b,c and d are real numbers with ad – bc = 1.

(Def) Let SL\_{2} (\mathbb{R}) denote the group of all 2 \times 2 matrices with real entreis and determinant 1, namely SL\_{2}(\mathb{R}) = {M = \begin{pmatrix}a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} and det(M) = ad-bc = 1}. This group is called the special linear group.

Given a matrix M \in SL\_{2}(\mathbb{R}) we define the mapping f\_{M} by f\_{M}(z) = \frac{az+b}{cz+d}.

(Thm 2.4.) Every automorphism of \mathbb{H} takes the form f\_{M} for some M \in SL\_{2}(\mathbb{R}) . Conversely, every map of this form is an automorphism of \mathbb{H}.

(Def) Since the two matrices M and -M give rise to the same function f\_{M} = f\_{-M}, if we identify the two matrices M and -M, then we obtain a new group PSL\_{2}(\mathbb{R}) called the projective special linear group; which is isomorphic with Aut(\mathbb{H}) .

(subsection 3) The Riemann mapping theorem

(subsubsection 3.1.) Necessary conditions and statement of the theorem

(Def) Call a subset \Omega of \mathbb{C} proper if it is non-empty and not the whole of \mathbb{C} .

(Riemann mapping theorem) (Thm 3.1.) Suppose \Omega is proper and simply connected. If z\_{0} \in \Omega, then there exists a unique conformal map F : \Omega \to \mathbb{D} s.t. F(z\_{0}) = 0 and F’(z\_{0}) >0.

(Cor 3.2.) Any two proper simply connected open subsets in \mathbb{C} are comformally equivalent.

(subsubsection 3.2.) Montel’s theorem

(Def) Let \Omega be an open subset of \mathbb{C}. A family \mathcal{F} of holomorphic functions on \Omega is said to be normal if every sequence in \mathcal{F} has a subsequence that converges uniformly on every compact subset of \Omega.

The family \mathcal{F} is said to be uniformly bounded on compact subsets of \Omega if for each compact set K \subset \Omega there exists B >0 s.t. |f(z)| \le B for all z \in K and f \in \mathcal{F}.

The family \mathcal{F} is equicontinuous on a compact set K if for every \epsilon >0 there exists \delta >0 s.t. whenever z , w \in K and |z-w| < \delta then |f(z)\_f(w)| < \epsilon for all f \in \mathcal{F}.

(Thm 3.3.) Suppose \mathcal{F} is a family of holomorphic functions on \Omega that is uniformly bounded on compact subsets of \Omega. Then

\mathcal{F} is equicontinuous on every compact subset of \Omega.

\mathcal{F} is a normal family.

(Def) A sequence {K\_{l}}\_{l=1}^{\infty} of compact subsets of \Omega is called an exhaustion if

K\_{l} is contained in the interor of K\_{l+1} for all l = 1,2,…

Any compact set K \subset \Omega is contained in K\_{l} for some l. In particular \Omega = \bigcup\_{l=1}^{\infty} K\_{l}.

(Lem 3.4.) Any open set \Omega in the complex plane has an exhaustion.

(Prop 3.5.) If \Omega is a connected open subset of \mathbb{C} and {f\_{n}} a sequence of injective holomorphic functions on \Omega that converges uniformly on every compact subset of \Omega to a holomorphic function f, then f is either injective or constant.

(subsubsection 3.3.) Proof of Riemann mapping theorem

(Def) \Omega is holomorphically simply connected if for any holomorphic function f in \Omega and any closed curve \gamma in \Omega, we have \int\_{\gamma} f(z) dz = 0.

(subsection 4) Conformal mappings onto polygons

(subsubsection 4.2.) The Schwarz-Christoffel integral

(Def) The general Schwarz-Christoffel integral by S(z) = \int\_{0}^{z} \frac{d\zeta}{(\zeta – A\_{1})^{\beta\_{1}}\cdot(\zeta – A\_{n})^{\beta\_{n}}}. Here A\_{1} < A\_{2} < \cdot < A\_{n} are n distinct points on the real axis arranges in increasing order. The exponents \beta\_{k} will be assumed to satisfy the conditions \beta\_{k} <1 for each k and 1 < \sum\_{k=1}^{n} \beta\_{k}.

The integrand is defined as follows : (z-A\_{k})^{\beta\_{k}} is that branch (Defined in the complex plane slit along the infinite ray {A\_{k} + iy : y \le 0}) which is positive when z = x is real and x > Z\_{k} . As a result (z-A\_{k})^{\beta\_{k}} = \begin{cases} (x-A\_{k})^{\beta\_{k}} & if x is real and x > A\_{k} \\ |x – A\_{k}|^{\beta\_{k}} e^{i\pi \beta\_{k}} & if x is real and x < A\_{k}.\end{cases}

(Prop 4.1.) Suppose S(z) is given as above.

If \sum\_{k=1}^{n} \beta\_{k} = 2, and \mathfrak{p} denotes the polygon whose vertices are given (in order) by a\_{1}, …, a\_{n}, then S maps the real axis onto \mathfrak{p} – {a\_{\infty}}. The point a\_{\infty} lies on the segment [a\_{n}, a\_{1}] and is the image of the point at infinity. Moreover, the (interior) angle at the vertex a\_{k} is \alpha\_{k} \pi where \alpha\_{k} = 1- \beta\_{k}.

There is a similar conclusion when 1< \sum\_{k=1}^{n} \beta\_{k} <2 except now the image of the extended line is the polygon of n+1 sides with vertices a\_{1}, a\_{2},… , a\_{n} , a\_{\infty}. The angle at the vertex a\_{\infty} is \alpha\_{\infty} \pi with \alpha\_{\infty} = 1- \beta\_{\infty}, where \beta\_{\infty} = 2 - \sum\_{k=1}^{n} \beta\_{k}

(subsubsection 4.3.) Boundary behavior

(Def) a polygonal region P, namely a bounded, simply connected open set whose boundary is a polygonal line \mathfrak{p}. In this context, we always assume that the polygonal line is closed and we sometimes refer to \mathfrak{p} as a polygon.

(Thm 4.2.) If F : \mathbb{D} \to P is a conformal map, then F extends to a continuous bijection from the closure \bar{\mathbb{D}} of the disc to the closure \bar{P} of the polygonal region. In particular, F gives rise to a bijection from the boundary of the disc to the boundary polygon \mathfrak{p}.

(Lem 4.3.) For each 0 < r < 1/2, let C\_{r} denote the circle centered at z\_{0} of radius r. Suppose that for all sufficienctly small r we are given two points z\_{r} and z’\_{r} in the unit disc that also lie on C\_{r}. If we let \rho(r) = |f(z\_{r}) – f(z’\_{r})| , then there exists a sequence {r\_{n}} of radii that tends to zero, and s.t. \lim\_{n \to \infty} \rho(r\_{n}) = 0.

(Lem 4.4.) Let z\_{0} be a point on the unit circle. Then F(z) tends to a limit as z approaches z\_{0} within the unit disc.

(Lem 4.5.) The conformal map F extends to a continuous function from the closure of the disc to the closure of the polygon.

(subsubseciton 4.4.) The mapping formula

(Def) Suppose P is a polygonal region bounded by a polygon \mathfrak{p} whose vertices are ordered consecutively a\_{1},a\_{2},…,a\_{n} and with n \ge 3. We denote by \pi \alpha\_{k} the interior angle at a\_{k}, and define the exterior angle \pi \beta\_{k} by \alpha\_{k} + \beta\_{k} = 1. A simple geometric argument provides \sum\_{k=1}^{n} \beta\_{k} = 2.

(Thm 4.6.) There exist complex numbers c\_{1} and c\_{2} so that the conformal map F of \mathbb{H} to P is given by F(z) = c\_{1}S(z) + c\_{2} where S is the Schwarz-Christoffel integral introduced before.

(Thm 4.7.) If F is a conformal map from the upper half-plane to the polygonal region P and maps the points A\_{1},…,A\_{n-1}, \infty to the vertices of \mathfrak{p}, then there exists constants C\_{1} and C\_{2} s.t. F(z) = C\_{1} \int\_{0}^{z} \frac{d\zeta}{(\zeta – A\_{1})^{\beta\_{1}}\cdot(\zeta – A\_{n-1})^{\beta\_{n-1}}} + C\_{2}.

(subsubsection 4.5.) Return to elliptic integrals

(Def) The elliptic integral I(z) = \int\_{0}^{z} \frac{d \zeta}{[(1-\zeta^{2})(1-k^{2}\zeta^{2}]^{1/2}} with 0 < k < 1.

(prop)it mapped the real axis to the rectangle R with vertices -K, K, K + i K’, and -K + iK’.

This mapping is a conformal mapping of \mathbb{H} to the interior of R.

(section 9) An introduction to Elliptic Functions

(subsection 1) Elliptic functions

(Def) meromorphic functions f on \mathbb{C} that have two periods; there are two non-zaro complex numbers \omega\_{1} and \omega\_{2} s.t. f(z + w\_{1}) = f(z) and f(z+w\_{2}) = f(z) for all z \in \mathbb{C} . A function with two periods is said to be doubly periodic.

Assume f is a meromorphic function on \mathbb{C} with periods 1 and \tau where Im(\tau) >0. Successive applications of the periodicity conditions yield f(z + n + m \tau) = f(z) for all integers n,m and all z \in \mathbb{C}, and it is therefore natural to consider the lattice in \mathbb{C} defined by \Lambda = {n + m\tau : n , m \in \mathbb{Z}} . We say that 1 and \tau generate \Lambda.

Associated to the lattice \Lambda is the fundamental parallelogram defined by P\_{0} = {z \in \mathbb{C} : z = a + b \tau where 0 \le a < 1 and 0 \le b < 1}.

(Def) Two complex numbers z and w are congruent modulo \Lambda if z = w + n + m \tau for some n , m \in \mathbb{Z}, and we write z \sim w. in other words, z and w differ by a point in the lattice, z – w \in \Lambda.

(Def) A period parallelogram P is any translate of the fundamental parallelogram , P = P\_{0} + h with h \in \mathbb{C}.

(Prop) \Lambda and P\_{0} give rise to a covering (or tiling) of the complex plane \mathbb{C} = \bigcup\_{n , m \in \mathbb{Z}} (n + m\tau + P\_{0}), and this union is disjoint.

(Prop 1.1.) Suppose f is a meromorphic function with two periods 1 and \tau which generate the lattice \Lambda. Then :

Every point in \mathbb{C} is congruent to a unique point in the fundamental parallelogram.

Every point in \mathbb{C} is congruent to a unique point in the fundamental parallelogram.

The lattice \Lambda provides a disjoint covering of the complex plane, in the sense of above (prop).

The function f is completely determined by its values in any period parallelograms.

(subsubsection 1.1.) Liouville’s theorems

(Thm 1.2.) An entire doubly periodic function is constant.

(Def) A non-constant doubly periodic meromorphic function is called an elliptic function.

(Thm 1.3.) The total number of poles of an elliptic function in P\_{0} is always \ge 2.

(Thm 1.4.) Every elliptic function of order m has m zeros in P\_{0}.

(subsubsection 1.2.) The Weierstrass \varphi function

(Def) Let \Lambda^{\*} denote the lattice minus the origin, that is, \Lambda^{\*} = \Lambda – {(0,0)}, and consider instead the following series : \frac{1}{z^{2}} + \sum\_{\omega \in \Lambda^{\*}} [\frac{1}{(z+\omega)^{2}} - \frac{1}{\omega^{2}}].

(Lem 1.5.) The two series \sum\_{(n,m) \neq (0,0)} \frac{1}{(|n| + |m|)^{r}} and \sum\_{n+m \tau \in \Lambda^{\*}} \frac{1}{|n+m\tau|^{r}} converge if r > 2.

(Def) Define Weierstrass \varphi function, which is given by the series \frac{1}{z^{2}} + \sum\_{\omega \in \Lambda^{\*}} [\frac{1}{(z+\omega)^{2}} - \frac{1}{\omega^{2}}]. = \frac{1}{z^{2}} + \sum\_{(n,m) \neq (0,0)}[\frac{1}{(z + n + m \tau}^{2} - \frac{1}{(n+m\tau)^{2}}].

\varphi is a meromorphic function with double poles at the lattice points.

(Thm 1.6.) The function \varphi is an elliptic function that has periods 1 and \tau, and double poles at the lattice points.

(Def) Since \varphi’ is elliptic and has order 3, the three points 1/2, \tau/2, and (1+\tau)/2 (which are called the half-periods) are the only roots of \varphi’ in the fundamental parallelogram, and they have multiplicity 1. Therefore, if we define \varphi(1/2) = e\_{1}, \varphi(\tau/2) = e\_{2} and \varphi(\frac{1+\tau}{2}) = e\_{3}

(Thm 1.7.) The function (\varphi’)^{2} is the cubic polynomial in \varphi (\varphi’)^{2} = 4(\varphi – e\_{1}) (\varphi -e\_{2}) (\varphi – e\_{3}) .

(Thm 1.8.) Every elliptic function f with periods 1 and \tau is a rational function of \varphi and \varphi ‘.

(Lem 1.9.) Every even elliptic function F with periods 1 and \tau is a rational function of \varphi.

(subsection 2) The modular character of elliptic functions and Eisenstein series

(Def) Consider the group of transformations of the upper half-plane Im(\tau) >0, generated by the two transformations \tau \mapsto \tau +1 and \tau \mapsto -1/\tau. This group is called the modular group.

(subsubsection 2.1.) Eisenstein series

(Def) The Eisenstein series of order k is defined by E\_{k}(\tau) = \sum\_{(n,m) \neq (0,0)} \frac{1}{(n+m\tau)^{k}}, whenever k is an integer \ge 3 and \tau is a complex number with Im(\tau) >0. If \Lambda is the lattice generated by 1 and \tau, and if we write \omega = n + m \tau, then another expression for the Eisenstein series is \sum\_{\omega \in \Lambda^{\*}} 1/\omega^{k}.

(Thm 2.1.) Eisenstein series have the following properties :

The series E\_{k}(\tau) converges if k \ge 3, and is holomorphic in the upper half-plane.

E\_{k}(\tau) = 0 if k is odd.

E\_{k}(\tau) satisfies the following transformation relations:

E\_{k}(\tau +1) = E\_{k}(\tau) and E\_{k}(\tau) = \tau^{-k} E\_{k}(-1/\tau).

The last property is sometimes referred to as the modular character of the Eisenstein series.

(Thm 2.2.) For z near 0, we have \varphi(z) = \frac{1}{z^{2}} + 3E\_{4}z^{2} + 5E\_{6}z^{4} + \cdot = \frac{1}{z^{2}} + \sum\_{k=1}^{\infty} (2k+1) E\_{2k+2} z^{2k}.

(Cor 2.3.) If g\_{2} = 60E\_{4} and g\_{3} = 140E\_{6}, then (\varphi ‘)^{2} = 4 \varphi^{3} – g\_{2} \varphi – g\_{3}.

(subsubsection 2.2.) Eisenstein series and divisor functions

(Lem 2.4.) If k \ge 2 and Im(\tau) >0, then \sum\_{n=-\infty}^{\infty} \frac{1}{(n+\tau)^{k}} = \frac{(-2\pi i)^{k}}{(k-1)!} \sum\_{l=1}^{\infty} l^{k-1} e^{2\pi i \tau l}.

(Def) The divisor function \sigma\_{l}(r) that arises here is defines as the sum of the l^{th} powers of the divisors of r, that is , \sigma\_{l}(r) = \sum\_{d|r} d^{l} .

(Thm 2.5.) If k \ge 4 is even, and Im(\tau) >0, then E\_{k}(\tau) = 2 \zeta(k) + \frac{2(-1)^{k/2} (2\pi)^{k}}{(k-1)!}\sum\_{r=1}^{\infty} \sigma\_{k-1} (r) e^{2\pi i \tau r}.

(Cor 2.6.) The double sum defining F converges in the indicated order. We have F(\tau) = 2 \zeta(2) – 8 \pi^{2} \sum\_{r=1}^{\infty} \sigma(r) e^{2\pi i r \tau}, where \sigma(r) = \sum\_{d|r} d is the sum of the divisors of r.

(section 10) Applications of Theta functions

(subsection 1) Product formula for the Jacobi theta function

(Def) Jacobi’s theta function is defined for z \in \mathbb{C} and \tau \in \mathbb{H} by \Theta (z|\tau) = \sum\_{n = - \infty}^{\infty} = \sum\_{n = - \infty}^{\infty} e^{\pi i n^{2} \tau} e^{2 \pi i n z}.

Two significant special cases are \theta(\tau) and \vartheta(t), which are defined by \theta(\tau) = \sum\_{n = -\infty}^{\infty} e^{\pi i n^{2} \tau} , \tau \in \mathbb{H}, \vartheta(t) = \sum\_{n = -\infty}^{\infty} e^{-\pi n^{2} t}, t>0.

For example, heat kernel was given by H\_{t}(x) = \sum\_{n = - \infty}^{\infty} = \sum\_{n = - \infty}^{\infty} e^{-4\pi^{2} n^{2} t} e^{2 \pi i n x}, and therefore H\_{t}(x) = \Theta(x|4\pi i t).

(Prop 1.1.) The function \Theta satisfies the following properties :

\Theta is entire in z \in \mathbb{C} and holomorphic in \tau \in \mathbb{H}.

\Theta(z+1|\tau) = \Theta(z | \tau).

\Theta(z + \tau | \tau) = \Theta(z|\tau) e^{-\pi i \tau} e^{-2\pi i z}

\Theta(z|\tau) = 0 whenever z = 1/2 + \tau/2 + n + m\tau and n,m \in \mathbb{Z}.

(Def) A product \prod(z|\tau) that enjoys the same structural properties as \Theta(z|\tau) as a function of z. This product is defined for z \in \mathbb{C} and \tau \in \mathbb{H} by \prod(z|\tau) = \prod\_{n=1}^{\infty} (1-q^{2n}) (1+ q^{2n-1}e^{2\pi i z}) (1+ q^{2n-1}e^{-2\pi i z}) , where we have used the notation that is standard in the subject, namely q = e^{\pi i \tau}. The function \prod(z|\tau) is sometimes referred to as the triple-product.

(Prop 1.2.) The function \prod(z|\tau) satisfies the following properties :

\prod(z,\tau) is entire in z \in \mathbb{C} and holomorphic for \tau \in \mathbb{H}.

\prod(z+1|\tau) = \prod(z | \tau).

\prod(z + \tau|\tau) = \prod(z|\tau) e^{-\pi i \tau} e^{-2\pi i z}.

\prod(z|\tau) = 0 whenever z = 1/2 + \tau/2 + n + m\tau and n , m \in \mathbb{Z}. Moreover, these points are simple zeros of \prod(\cdot | \tau) , and \prod(\cdot | \tau) has no other zeros.

(Product formula) (Thm 1.3.) For all z \in \mathbb{C} and \tau \in \mathbb{H} we have the identity \Theta(z|\tau) = \prod(z | \tau).

(Cor 1.4.) If Im(\tau)>0 and q = e^{\pi i \tau}, then \theta(\tau) = \prod\_{n=1}^{\infty} (1-q^{2n}) (1+ q^{2n-1})^{2}. Thus \theta(\tau) \neq 0 for \tau \in \mathbb{H}.

(Cor 1.5.) For each fixed \tau \in \mathbb{H}, the quotient (log \Theta(z|\tau))” = \frac{\Theta(z|\tau) \Theta”(z|\tau) – (\Theta’(z|\tau))^{2}}{\Theta(z|\tau)^{2}} is an elliptic function of order 2 with periods 1 and \tau, and with a double pole at z = 1/2 + \tau/2.

In the above, the primes ‘ denote differentiation with respect to the z variable.

(subsubsection 1.1.) Further transformation laws

(Thm 1.6.) If \tau \in \mathbb{H}, then \Theta(z|-1/\tau) = \sqrt{\frac{\tau}{i}} e^{\pi i \tau z^{2}} \Theta(z\tau | \tau) for all z \in \mathbb{C}. Here \sqrt{\frac{\tau}{i}} denotes the branch of the square root defined on the upper half-plane, that is positive when \tau = it, t >0.

(Cor 1.7.) If Im(\tau) >0, then \theta(-1/\tau) = \sqrt{\frac{\tau}{i}}\theta(\tau).

(Cor 1.8.) If \tau \in \mathbb{H}, then \theta(1-1/\tau) = \sqrt{\frac{\tau}{i}} \sum\_{n=-\infty}^{\infty} e^{\pi i (n+1/2)^{2} \tau} = \sqrt{\frac{\tau}{i}}( 2 e^{\pi i \tau/4} + \cdots). The second identity means that \theta(1-1/\tau) \sim \sqrt{\frac{\tau}{i}} 2e^{i\pi \tau/4} as Im(\tau) \to \infty.

(Def) Dedekind eta function, which is defined for Im(\tau) >0 by \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod\_{n=1}^{\infty} (1- e^{2\pi i n \tau}).

(prop 1.9.) If Im(\tau) >0, then \eta(-1/\tau) = \sqrt{\tau/i} \eta(\tau).

(subsection 2) Generating functions

(Def) Given a sequence {F\_{n}}\_{n=0}^{\infty}, its generating function F(x) = \sum\_{n=0}^{\infty} F\_{n}x^{n}.

(Def) The partition function is defined as follows : if n is a positive integer, we let p(n) denote the numbers of ways n can be written as a sum of positive integers. For instance, p(1) = 1, and p(2) = 2 since 2 = 2+0 = 1+1. We set p(0) = 1.

(Thm 2.1.) If |x| < 1, then \sum\_{n=1}^{\infty} p(n) x^{n} = \prod\_{k=1}^{\infty} \frac{1}{1-x^{k}}.

(Def) Let p\_{e,u}(n) denote the number of partitions of n into an even number of unequal parts, and p\_{0,u} (n ) the number of partitions of n into an odd number of unequal parts. Then , Euler proved that, unless n is a pentagonal number, one has p\_{e,u}(n) = p\_{o,u}(n) . By definition, pentagonal numbers are integers n of the form k(3k+1)/2, with k \in \mathbb{Z}.

(Prop 2.2.) \prod\_{n=1}^{\infty}(1-x^{n}) = \sum\_{k=-\infty}^{\infty} (-1)^{k}x^{\frac{k(3k+1)}{2}}

(subsection 3) The theorems about sum of squares

(subsubseciton 3.1.) The two-squares theorem

(Def) divisor functions : d\_{1}(n) denote the number of divisors of n of the form 4k +1, and d\_{3} (n) the number of divisors of n of the form 4k +3.

define r\_{2}(n) to be the number of ways n can be written as the sum of two squares, counting obvious repetitions; that is, r\_{2}(n) is the number of pairs (x,y) x, y \in \mathbb{Z}, so that n = x^{2} + y^{2} .

Define r\_{4}(n) to be the number of ways of expressing n as a sum of four squares.

(Thm 3.1.) If n \ge 1, then r\_{2}(n) = 4(d\_{1}(n) – d\_{3}(n)).

If n = p\_{1}^{\alpha\_{1}}\cdots p\_{r}^{\alpha\_{r}} is the prime factorization of n where p\_{1},…,p\_{r} are distinct, then : The positive integer n can be represented as the sum of two squares iff every prime p\_{j} of the form 4k + 3 that occurs in the factorization of n has an even exponent a\_{j}.

(Prop 3.2.) The identity r\_{2}(n) = 4(d\_{1}(n) – d\_{3}(n)) , n \ge 1, is equivalent to the identities \theta(\tau)^{2} = 2 \sum\_{n=-\infty}^{\infty}\frac{1}{q^{n} + q^{-n}} = 1 + 4 \sum\_{n=1}^{\infty} \frac{q^{n}}{1+q^{2n}}. whenever q = e^{\pi i \tau} and \tau \in \mathbb{H}.

(Prop 3.3.) The function \mathcal{C}(\tau) = \sum 1/cos(\pi n \tau) , defined in the upper half-plane, satisfies

\mathcal{C}(\tau +2) = \mathcal{C}(\tau).

\mathcal{C}(\tau) = (i/\tau)\mathcal{C}(-1/\tau).

\mathcal{C}(\tau) \to 1 as Im(\tau) \to \infty.

\mathcal{C}(1-1/\tau) \sim 4(\tau/i) e^{\pi i \tau/2} as Im(\tau) \to \infty.

Moreover, \theta(\tau)^{2} satisfies the same properties.

(Thm 3.4.) Suppose f is a holomorphic function in the upper half-plane that satisfies:

f(\tau +2) = f(\tau),

f(-1/\tau) = f(\tau),

f(\tau) is bounded,

then f is constant.

(Def) A subset of the closed upper half-plane, which is defined by \mathcal{F} = {\tau \in \bar{\mathbb{H}} : |Re(\tau)|\le 1 and |\tau| \ge 1}. The points corresponding to \tau = \mp 1 are called cusps.

(Lem 3.5.) Every point in the upper half-plane can be mapped into \mathcal{F} using repeatedly one or another of the following fractional linear transformations or their inverses: T\_{2} : \tau \mapsto \tau +2, S : \tau \mapsto -1/\tau.

For this reason, \mathcal{F} is called the fundamental domain for the group of transformations generated by T\_{2} and S.

(Def) We let G denote the group generated by T\_{2} and S. Since T\_{2} and S are fractional linear transformations, we may represent an element g \in G by a matrix g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} with the understanding that g(\tau) = \frac{a\tau + b}{c\tau + d}. Since the matrices representing T\_{2} and S have integer coefficients and determinant 1, the same is true for all matrices of elements in G. In particular, if \tau \in \mathbb{H}, then Im(g(\tau)) = \frac{Im(\tau)}{|c\tau + d|^{2}}.

(subsection 3.2.) The four-squares theorem

(Def) \sigma\_{1}^{\*}(n) equals the sum of divisors of n that are not divisible by 4.

(Thm 3.6.) Every positive integer is the sum of four squares, and moreover r\_{4}(n) = 8 \sigma\_{1}^{\*}(n) for all n \ge 1.

(Prop 3.7.) The assertion r\_{4}(n) = 8\signa\_{1}^{\*} (n) is equivalent to the identity \theta(\tau)^{4} = \frac{-1}{\pi^{2}} E\_{2}^{\*}(\tau), where \tau \in \mathbb{H}.

(Prop 3.8.) The function E\_{2}^{\*}(\tau) defined in the upper half-plane has the following properties :

E\_{2}^{\*}(\tau + 2) = E\_{2}^{\*}(\tau).

E\_{2}^{\*}(\tau) = -\tau^{-2} E\_{2}^{\*{}(-1/\tau).

E\_{2}^{\*}(\tau) \to -\pi^{2} as Im(\tau) \to \infty.

|E\_{2}^{\*} (1-1/\tau)| = O(|\tau^{2} e^{\pi i \tau}|) as Im(\tau) \to \infty.

moreover -\pi^{2}\theta^{4} has the same properties.

(Def) The forbidden Eisenstein series F and its reverse \tilde{F}, which is obtained from reversing the order of summation : F(\tau) = \sum\_{m} \sum\_{n} \frac{1}{(m\tau + n)^{2}} and \tilde{F} (\tau) = \sum\_{n} \sum\_{m} \frac{1}{(m\tau + n)^{2}}. In both cases, the term n = m = 0 is omitted.

(Lem 3.9.) The functions F and \tilde{F} satisfy :

F(-1/\tau) = \tau^{2} \tilde{F}(\tau),

F(\tau) - \tilde{F}(\tau) = 2\pi i/\tau.

F(-1/\tau) = \tau^{2}F(\tau) – 2\pi i \tau.

(Appendix A) Asymptotics

(subsection 1) Bessel functions

(Def) For all order \nu > -1/2, J\_{\nu}(s) = \frac{(s/2)^{\nu}}{\Gamma(\nu + 1/2) \Gamma(1/2)} \int\_{-1}^{1} e^{isx} (1-x^{2})^{\nu – 1/2} dx.

If we also write J\_{-1/2}(s) for \lim\_{v\to-1/2} J\_{\nu}(s) , it equals \sqrt{\frac{2}{\pi s}} cos s; . J\_{1/2}(s) = \sqrt{\frac{2}{\pi s}} sin s.

(Thm 1.1.) J\_{\nu}(s) = \sqrt{2}{\pi s} cos (s- \frac{\pi \nu}{2} - \frac{\pi}{4}) + O(s^{-3/2}) as s \to \infty.

(Prop 1.2.) Suppose a and m are fixed, with a >0 and m > -1. Then as s \to \infty \int\_{0}^{a} e^{-sx} x^{m} dx = s^{-m-1} \Gamma(m+1) + O(e^{-cs}), for some positive c.

(Prop 1.3.) Suppose a and m are fixed, with a >0 and -1<m<0. Then as |s| \to \infty with Re(s) \ge 0, \int\_{0}^{a} e^{-sx} x^{m} dx = s^{-m-1} \Gamma(m+1) + O(1/|s|). (Here s^{-m-1} is the branch of that function that is positive for s >0.)

(subsection 2) Laplace’s method ; Stirling’s formula

(Def) Consider \int\_{a}^{b} e^{-s\Phi(x)} \psi(x) dx where the phase \Phi is real-valued, and both it and the amplitude \psi are assumed for simplicity to be indefinitely differentiable. Our hypothesis regarding the minimum of \Phi is that there is an x\_{0} \in (a,b) so that \Phi’(x\_{0}) = 0, but \Phi”(x\_{0}) >0 throughout [a,b].

(Prop 2.1.) Under the above assumptions, with s >0 and s \to \infty, \int\_{a}^{b} e^{-s\Phi(x)}\psi(x) dx = e^{-s\Phi(x\_{0})} [\frac{A}{s^{1/2}} + O(\frac{1}{s})], where A = \sqrt(2\pi) \frac{\psi(x\_{0})}{(\Phi”(x\_{0})^{1/2}}.

(Prop 2.2.) With the same assumptions on \Phi and \psi, the relation of (prop 2.1.) continues to hold if |s| \to \infty with Re(s) \ge 0.

(Def) The special case of (Prop 2.2.) When s is purely imaginary, s = it, t \to \mp \infty , is often treated separately; the argument in this situation is usually referred to as the method of stationary phase. The points x\_{0} for which \Phi’(x\_{0}) = 0 are called the critical points.

(Thm 2.3.) If |s| \to \infty with s \in S\_{\delta} , then \Gamma(s) = e^{s log s} e^{-s} \frac{\sqrt{2\pi}}{s^{1/2}} (1+ O(\frac{1}{|s|^{1/2}}) ).

(subsection 3) The Airy function

(Def) The Airy function Ai is defined by Ai(s) = \frac{1}{2\pi} \int\_{-\infty}^{\infty} e^{i(x^{3}/3+sx)}dx , with s \in \mathbb{R}.

Ai”(s) = s Ai(s).

(Thm 3.1.) Suppose u >0. Then as u \to \infty,

Ai(-u) = \pi^{-1/2} u^{-1/4} cos(\frac{2}{3} u^{3/2} - \frac{\pi}{4})(1+O(1/u^{3/4})).

Ai(u) = \frac{1}{2\pi^{1/2}} u^{-1/4} e^{-\frac{2}{3} u^{3/2}} (1+ O(1/u^{3/4})).

(De) Assume F(z) is holomorphic. We seek a contour \Gamma so that

Im(F) = 0 on \Gamma.

Re(F) has a minimum on \Gamma at some point z\_{0}, and this function is non-degenerate in the sense that the second derivative of Re(F) along \Gamma is strictly positive at z\_{0}.

If as above, F”(z\_{0}) \neq 0, then there are two curves \Gamma\_{1} and \Gamma\_{2} passing through z\_{0} which are orthogonal, so that F|\_{\Gamma}, is real for i = 1,2, with Re(F) restricted to \Gamma\_{1} having a minimum at z\_{0}; and Re(F) restricted to \Gamma\_{2} having a maximum at z\_{0} . We try to deform out original contour of integration to \Gamma = \Gamma\_{1}. This approach is usually referred to as the method of steepest descent.

(subsection 4) The partition function

(Thm 4.1.) If p denotes the partition function, then

p(n)~ \frac{1}{4\sqrt{3} n} e^{Kn^{1/2}} as n \to \infty, where K = \pi \sqrt{\frac{2}{3}}.

A much more precise assertion is that p(n) = \frac{1}{2\pi \sqrt{2}} \frac{d}{dn} (\frac{e^{K(n-\frac{1}{24})^{1/2}}}{(n-\frac{1}{24})^{1/2}}) + O(e^{\frac{K}{2} n^{1/2}}).

(Appendix B) Simple Connectivity and Jordan Curve Theorem

(subsection 1) Equivalent formulations of simple connectivity

(Thm 1.1.) A region \Omega is holomorphically simply connected iff \Omega is simply connected.

(Thm 1.2.) If \Omega is a bounded region in \mathbb{C}, then \Omega is simply connected iff the complement of \Omega is connected.

(Def) The winding number of a closed curve \gamma around a point z \notin \gamma is W\_{\gamm}(z) = \frac{1}{2\pi i} \int\_{\gamma} \frac{d\zeta}{\zeta - z}. Sometimes, W\_{\gamma}(z) is also called the index of z with respect to \gamma.

(Lem 1.3.) Let \gamma be a closed curve in \mathbb{C}.

If z \notin \gamma, then W\_{\gamma}(z) \in \mathbb{Z}.

If z and w belong to the same open connected component in the complement of \gamma, then W\_{\gamma}(z) = W\_{\gamma}(w).

If z belongs to the unbounded connected component in the complement of \gamma, then W\_{\gamma}(z) = 0.

(Thm 1.4.) A bounded region \Omega is simply connected iff W\_{\gamma}(z) = 0 for any closed curve \gamma in \Omega and any point z not in \Omega.

(Lem 1.5.) Let w be any point in F\_{1}. Under the above assumptions. there exists a finite collection of closed squares \mathcal{Q} = {Q\_{1},…,Q\_{n}} that belong to a uniform grid \mathcal{G} of the plane, and are such that:

w belongs to the interior of Q\_{1}.

The interiors of Q\_{j} and Q\_{k} are disjoint when j \neq k.

F\_{1} is contained in the interior of \cup\_{j=1}^{n} Q\_{j}.

\cup\_{j=1}^{n} Q\_{j} is disjoint from F\_{2}.

The boundary of \cup\_{j=1}^{n} Q\_{j} lies entirely in \Omega, and consists of a finite union of disjoint simple closed polygonal curves.

(subsection 2) The Jordan curve theorem

(Thm 2.1.) Let \Gamma be curve in the plane that is simple and piecewise smooth. Then, the complement of \Gamma is an open connected set whose boundary is precisely \Gamma.

(Thm 2.2.) Let \Gamma be a curve in the plane which is simple, closed, and piecewise-smooth. Then, the complement of \Gamma consists of two disjoint connected open sets. Precisely one of these regions is bounded and simply connected; it is called the interior of \Gamma and denoted by \Omega. The other component is unbounded, called the exterior of \Gamma, and denoted by \mathcal{U}. Moreover, with the appropriate orientation for \Gamma, we have W\_{\Gamma}(z) = \begin{cases} 1 & if z \in \Omega \\ 0 & if z \in \mathcal{U}\end{cases}.

(Thm 2.3.) Suppose f is a function that is holomotphic in the interior \Omega of a simple closed curve \Gamma. Then \int\_{\eta} f(\zeta) d \zeta = 0 whenever \eta is any closed curve contained in \Omega.

(prop) recall that an arc-length parametrization \gamma for a smooth curve \Gamma\_{0} satisfies |\gamma’(t)| = 1 for all t. Every smooth curve has an arc-length parametrization.

(Lem 2.4.) Let \Gamma\_{0} be a simple smooth curve with an arc-length parametrization given by \gamma : [0,L] \to \mathbb{C}. For each real number \epsilon, Let \Gamma\_{\epsilon} be the continuous curve defined by the parametrization \gamma\_{\epsilon} (t) = \gamma(t) + i\epsilon \gamma’(t) , for 0 \le t \le L. Then, there exists \kappa\_{1}>0 so that \Gamma\_{0} \cap \Gamma\_{\epsilon} = \empty whenever 0 < |\epsilon| < \kappa\_{1}.

(Lem 2.5.) Suppose z is a point which does not belong to the smooth curve \Gamma\_{0}, but that is closer to an interior point of the curve than to either of its end-points. Then z belongs to \Gamma\_{\epsilon} for some \epsilon \neq 0. More precisely, if z\_{0} \in \Gamma\_{0} is closest to z and z\_{0} = \gamma(t\_{0}) for some t\_{0} in the open interval (0,L) , then z = \gamma(t\_{0}) + i\epsilon \gamma’(t\_{0}) for some \epsilon \neq 0.

(prop 2.6.) Let A and B denote the two end-points of a simple smooth curve \Gamma\_{0}, and suppose that K is a compact set that satisfies either \Gamma\_{0} \cap K = \empty or \Gamma\_{0} \cap K = A \cup B. If z \notin \Gamma\_{0} and w \notin \Gamma\_{0} lie on the same side of \Gamma\_{0}, and are closer to interor points of \Gamma\_{0} than they are to K or to the end-points of \Gamma\_{0}, then z and w can be joined by a continuous curve that lies entirely in the complement of K \cup \Gamma\_{0}.

(Lem 2.7.) Let \Gamma\_{0} be a simple smooth curve. There exists \kappa\_{2} >0 so that the set N, which consists of points of the form z = \gamma(L) + \epsilon e^{i\theta} \gamma’(L) with -\pi/2 \ge \theta \ge \pi/2 and 0 < \epsilon < \kappa\_{2} , is disjoint from \Gamma\_{0}.

(Prop 2.8.) Let A denote an end-point of the simple smooth curve \Gamma\_{0}, and suppose that K is a compact set that satisfies either \Gamma\_{0} \cap K = \empty or \Gamma\_{0} \cap K = A. If z \notin \Gamma\_{0} and w \notin \Gamma\_{0} are closer to inerior points of \Gamma\_{0} than they are to K or to the end-points of \Gamma\_{0}, then z and w can be joined by a continuous curve that lies entirely in the complement of \Gamma\_{0} \cap K.

(Thm 2.9.) If a function f is holomorphic in an open set that contains a simple closed piecewise-smooth curve \Gamma and its interior, then \int\_{\Gamma} f = 0.

(Lem 2.10) Let \gamma : [0,1] \to \mathbb{C} be a simple smooth curve. Then, for all sufficiently small \delta >0 the circle C\_{\delta} centered at \gamma(0) and of radius \delta intersects \gamma in precisely one point.